Jonas Kremer WS 2017/2018

Probability theory

Exercise Sheet 2

Exercise 1 *(4 Points)*
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\Omega', \mathcal{F}')$ a measurable space, and $\varphi : \Omega \to \Omega'$ measurable. We define the image measure $\mathbb{P} \circ \varphi^{-1}$ on $(\Omega', \mathcal{F}')$ by

$$\left(\mathbb{P} \circ \varphi^{-1}\right)(A) = \mathbb{P}\left(\varphi^{-1}(A)\right), \quad A \in \mathcal{F}'.$$

(a) Show that $(\Omega', \mathcal{F}', \mathbb{P} \circ \varphi^{-1})$ is a probability space.

(b) Let $f : \Omega' \to \mathbb{R}$ be measurable. Show that f is integrable with respect to $\mathbb{P} \circ \varphi^{-1}$ if and only if $f \circ \varphi$ is integrable with respect to $\mathbb{P}$. Moreover, show that in this case, we have

$$\mathbb{E}\left[f\left(\varphi(\omega)\right)\right] = \int_{\Omega'} f(\omega') \left(\mathbb{P} \circ \varphi^{-1}\right)(\mathrm{d}\omega').$$

Exercise 2 *(4 Points)*
Suppose $X_n, X \in \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ and $Y_n,\ Y \in \mathcal{L}^q(\Omega, \mathcal{F}, \mathbb{P})$ with $p, q \in [1, \infty]$ such that

$$\|X_n - X\|_{\mathcal{L}^p} \longrightarrow 0 \text{ and } \|Y_n - Y\|_{\mathcal{L}^q} \longrightarrow 0, \quad n \to \infty.$$

Prove that $\|X_n Y_n - XY\|_{\mathcal{L}^1} \longrightarrow 0$ as $n \to \infty$, provided $1/p + 1/q = 1$.

Exercise 3 *(4 Points)*
Let $\Omega_1 = \Omega_2 = \mathbb{N}$, $\mathcal{F}_1 = \mathcal{F}_2 = 2^{\mathbb{N}}$, and $\mu_1 = \mu_2 = \mu$ be the counting measure

$$\mu(A) = \sum_{n \in \mathbb{N}} \delta_n(A), \quad A \subset \mathbb{N}.$$

Let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be given by

$$f(n,m) = \begin{cases} n, & n = m \\ -n, & m = n+1 \\ 0, & \text{otherwise} \end{cases}.$$

Show that

$$\int_{\Omega_1} \left(\int_{\Omega_2} f(n,m) \mathrm{d}\mu_2(m) \right) \mathrm{d}\mu_1(n) = 0,$$
$$\int_{\Omega_2} \left(\int_{\Omega_1} f(n,m) \mathrm{d}\mu_1(n) \right) \mathrm{d}\mu_2(m) = \infty.$$

Does this contradict Fubini's theorem?

Exercise 4 *(4 Points, talk)*
Look up the Theorem of Radon-Nikodym for probability measures. Find and prepare two examples for the particular case of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.